Law of Total Probability

\[ P(A) = P(A \cap B) + P(A \cap B') = P(B) \cdot P(A \mid B) + P(B') \cdot P(A \mid B') \]

In general,

\[ P(A) = \sum_{i=1}^{m} P(B_i) \cdot P(A \mid B_i) \]

Example 1:

Based on past data, it is believed that the proportion of individuals who have COVID-19 in Champaign-Urbana area is 0.025. A test for COVID-19 is positive in 94% of the people who have the virus and in 4% of the people who do not.

a) Find the probability of receiving a positive result from this COVID-19 test.

\[ P(C) = 0.025 \]

\[ P(+ \mid C) = 0.94 \]

\[ P(+ \mid C') = 0.04 \]

\[ P(+) = P(+ \cap C) + P(+ \cap C') \]

\[ = P(C) \cdot P(+ \mid C) + P(C') \cdot P(+ \mid C') \]

\[ = 0.025 \times 0.94 + (1 - 0.025) \times 0.04 \]

\[ = 0.0625 \]

b) If a person received a positive result from this test, what is the probability that he/she has the virus?

\[ P(C \mid +) = \frac{P(C \cap +)}{P(+)} = \frac{0.0235}{0.0625} = 0.376 \]
Bayes’ Theorem:

\[ P(B | A) = \frac{P(A) \cdot P(B | A)}{P(A)} = \frac{P(A) \cdot P(B | A)}{P(B) \cdot P(A | B) + P(B') \cdot P(A | B')} \]

\[ P(C | +) = \frac{P(C) \cdot P(+ | C)}{P(C) \cdot P(+ | C) + P(C') \cdot P(+ | C')} = \frac{0.025 \cdot 0.94}{0.025 \cdot 0.94 + 0.975 \cdot 0.04} = 0.376 \]

c) If a person received a negative result from this test, what is the probability that he/she actually doesn’t have the virus?

\[ P(C' | -) = \frac{P(C') \cdot P(- | C')}{P(C') \cdot P(- | C') + P(C) \cdot P(- | C)} = \frac{0.975 \cdot (1-0.04)}{0.975 \cdot (1-0.04) + 0.025 \cdot (1-0.94)} = 0.9984 \]

Example 2:

Company A is developing a new pregnancy test. Based on their experiments, the test gives the correct result about 88% of the time, meaning the test result is positive when the person is pregnant, it is negative when the person is not pregnant. Suppose that 55% of the women who take the test are pregnant.

What is the probability that the person is actually pregnant when their test result is negative?

- \( A \) = pregnant \( P(A) = 0.55 \Rightarrow P(A') = 0.45 \)
  \[ P(+ | A) = 0.88 \quad P(- | A') = 0.88 \]
  \[ P(A | -) = \frac{P(A) \cdot P(- | A)}{P(A) \cdot P(- | A) + P(A') \cdot P(- | A')} = \frac{0.55 \cdot 0.12}{0.55 \cdot 0.12 + 0.45 \cdot 0.88} = \frac{1}{7} \]

<table>
<thead>
<tr>
<th></th>
<th>+</th>
<th>-</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>121</td>
<td>33</td>
<td>0.55</td>
</tr>
<tr>
<td>( A' )</td>
<td>27</td>
<td>99</td>
<td>0.45</td>
</tr>
<tr>
<td></td>
<td>250</td>
<td>500</td>
<td></td>
</tr>
<tr>
<td></td>
<td>250</td>
<td>500</td>
<td>1</td>
</tr>
</tbody>
</table>

- \( P(A \cap +) = P(A) \cdot P(+ | A) = 0.55 \cdot 0.88 = \frac{121}{250} \)