Confidence Interval for Population Mean

A confidence interval for the population mean $\mu$ with confidence level $(1 - \alpha)100\%$:

- **$\sigma$ is known:**
  \[
  \bar{x} \pm z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}
  \]

- **$\sigma$ is unknown:**
  \[
  \bar{x} \pm t_{\alpha/2} \cdot \frac{s}{\sqrt{n}}
  \]

**Rules for computing CIs:**

- **Case 1:** $\sigma$ is known
  - Normal population
    \[
    \bar{x} \pm z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}
    \]
  - Unknown population
    \[
    \bar{x} \pm t_{\alpha/2} \cdot \frac{s}{\sqrt{n}}
    \]
  - Small $n$ (<30)
    \[
    \bar{x} - \mu \sim N(0, \frac{\sigma}{\sqrt{n}})
    \]

- **Case 2:** $\sigma$ is unknown
  - Normal population
    \[
    \bar{x} - \mu \sim t(\text{df} = n-1)
    \]
  - Unknown population
    \[
    \bar{x} - \mu \sim t(\text{df} = n-1)
    \]
  - Small $n$ (<30)
    \[
    \bar{x} - \mu \sim t(\text{df} = n-1)
    \]
  - Large $n$
    \[
    \bar{x} - \mu \sim z
    \]
Example 1:
Suppose the lifetime of a particular brand of light bulbs is normally distributed with standard deviation of \( \sigma = 75 \) hours and unknown mean.

a) Suppose the sample average lifetime of \( n = 49 \) bulbs is \( \bar{x} = 843 \) hours. Construct a 95% confidence interval for the overall average lifetime for light bulbs of this brand.

\[
\bar{x} = 75 \quad n = 49 \quad \bar{x} = 843 \quad \alpha = 0.05
\]

A 95% CI for \( \mu \):

\[
\bar{x} \pm z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}} = 843 \pm \text{norm.ppf}(q=0.975) \cdot \frac{75}{\sqrt{49}} \approx (822.0004, 863.9996)
\]

b) Construct a 90% confidence interval for the overall average lifetime for light bulbs of this brand.

\( \alpha = 0.1 \)

A 90% CI for \( \mu \):

\[
\bar{x} \pm z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}} = 843 \pm \text{norm.ppf}(q=0.95) \times \frac{75}{\sqrt{49}} \approx (825.3766, 860.6234)
\]

c) Construct a 99% confidence lower bound for the overall average lifetime for light bulbs of this brand.

\( \alpha = 0.01 \)

A 99% conf LB for \( \mu \):

\[
\bar{x} - z_{\alpha} \cdot \frac{\sigma}{\sqrt{n}} = 843 - \text{norm.ppf}(1-0.99) \times \frac{75}{\sqrt{49}} \approx 818.0748
\]

**Lower Bound:**

A \((1 - \alpha)100\%\) lower bound for \( \mu \):

- \( \sigma \) is known: \( \bar{x} - z_{\alpha} \cdot \frac{\sigma}{\sqrt{n}} \)
- \( \sigma \) is unknown: \( \bar{x} - t_{\alpha} \cdot \frac{s}{\sqrt{n}} \)

**Upper Bound:**

A \((1 - \alpha)100\%\) upper bound for \( \mu \):

- \( \sigma \) is known: \( \bar{x} + z_{\alpha} \cdot \frac{\sigma}{\sqrt{n}} \)
- \( \sigma \) is unknown: \( \bar{x} + t_{\alpha} \cdot \frac{s}{\sqrt{n}} \)
Example 2: A manufacturer of TV sets wants to find the average selling price of a particular model. A random sample of 25 different stores gives the mean selling price as $342 with a sample standard deviation of $14. Assume the prices are normally distributed. Construct a 95% confidence interval for the mean selling price of the TV model.

\[ ar{x} = 0.05 \]

A 95% CI for \( \mu \):
\[ \bar{x} \pm t_{\alpha/2} \frac{s}{\sqrt{n}} = 342 \pm t_{24} \cdot 14 \sqrt{25} \]
\[ \Rightarrow (366.2211, 347.7789) \]

Sample Size Calculation
The minimum required sample size in estimating the population mean \( \mu \) to within \( \varepsilon \) with \( (1 - \alpha)100\% \) confidence is
\[ n = \left[ \frac{Z_{\alpha/2} \cdot \sigma}{\varepsilon} \right]^2 \]
\[ \text{Always round } n \text{ up!} \]

Example 3: How many test runs of an automobile are required for determining its average miles-per-gallon rating on the highway to within 0.5 miles per gallon with 95% confidence, if a guess is that the variance of the population of miles per gallon is about 6.25 miles? \[ \Rightarrow \varepsilon = 0.5 \]
\[ \Rightarrow \alpha = 0.05 \]
\[ \Rightarrow \sigma = 0.5 \] \[ \Rightarrow \sigma^2 = 6.25 \]
\[ n = \left[ \frac{Z_{\alpha/2} \cdot \sigma}{\varepsilon} \right]^2 = \left[ \frac{\text{norm. ppf} \left(1 - \frac{0.05}{2}, \sqrt{6.25}\right)}{0.5} \right]^2 \approx 96.03 \]
\[ \Rightarrow n = 97 \]

To-do:
- Finish Lab 09, commit and push the lab using git commands!
- Get started with HW 8 on Prairie Learn!